UNCERTAINTY PRINCIPLES:

When elementary texts discuss the Uncertainty Relations of Heisenberg in early chapters, they suddenly drag in advanced topics that have not been used or explained before, such as properties of Fourier transforms, the operator form of quantum physics, non-commuting Hermitian operators, expectation values, statistical measures of uncertainty, and so on. [You will learn about all of these concepts in Physics 373.] However, a very simple qualitative discussion is possible, just based on the de Broglie relation $p = h/\lambda$. We just have to decide what $\lambda$ actually means in quantum physics, which is in reality entirely expressed in terms of probability amplitudes and distributions. This is a lot harder than it sounds, which is why most texts suddenly jump to advanced material which has not been introduced before. The very basic idea is that the wavelength is related to how much the system’s position uncertainty is. If the probability distribution is just a bump, the position is well known within the width of the bump. But if the probability distribution happens to have wiggles, like the absolute value of an extended wave form squared, the position is uncertain... the particle could be in any one of those bumps.
So to the extent $\lambda$ is well defined, the position is uncertain. Let's consider a one-dimensional case. Then, as we argued, $\lambda$ is proportional to $\Delta x$. Therefore there is a corresponding momentum uncertainty, $\Delta p_x \sim h/\Delta x$. If we were to go through the far more advanced analysis, we would wind up with $\Delta p_x \geq h/(4\pi \Delta x)$. [A huge amount of work just to get a factor of $4\pi$!]

This is of course usually written as

$$\Delta p_x \times \Delta x \geq \frac{h}{2}.$$ 

Now look at the other famous equation $E = h\nu$, or using the angular frequency $\omega = 2\pi \nu$, just $E = \hbar \omega$. A definite value of $E$ would correspond to a definite value of $\omega$. But the probability distributions in general are bumps or a series of bumps, not at all like a full wave form. Since $\lambda$ is uncertain, so is $\omega$ and therefore $E$. We can write $\Delta E \sim \hbar \Delta \omega$. But for a wave in general, the phase speed is $v_p = \omega/k$, where $k$ is the wave number $k = 2\pi/\lambda$. And of course the period of a wave is $T = 1/\nu = 2\pi/\omega$, so $\omega = 2\pi/T$. Since we will not in general see an actual wave, we will not see a full oscillation, so $T \simeq \Delta t$, and we wind up with $\Delta E \Delta t \sim h$. As before, a more detailed
analysis using advanced ideas of quantum physics and statistics gives us

\[ \Delta E \times \Delta t \geq \frac{\hbar}{2}. \]

One thing I notice in reading over various physics textbooks presenting quantum ideas at various levels is near-total confusion about the "wave" aspects of quantum physics. For example, it is often said that the uncertainty relations are a result of the wave aspects of quantum physics. While there is certainly a bit of truth in this statement, given the importance Fourier transforms turn out to have in quantum physics, the uncertainty relations exist because there are no wavelike solutions, in general, to quantum problems. For example, imagine a probability amplitude which is wavelike over all space, so that the wavelength and therefore the momentum of the state described are well-defined... no uncertainty! The uncertainty results from the fact that the amplitudes in general are rarely wavelike, so that both positions and momenta are uncertain to a significant degree!

In 373 you will learn in more detail where various uncertainty relations in physics come from. Observable quantities are represented by operators, and when
two such operators do not commute... give different results when applied in different order... there is a corresponding uncertainty relation for the observables.

In mathematical statistics the uncertainty in the value of a variable $A$ is precisely defined by $(\Delta A)^2 = \langle A^2 \rangle - (\langle A \rangle)^2$. In quantum physics, the quantity $\langle A \rangle$ is called the expectation value of $A$ (for a given state $\Psi$) and you will learn how to calculate it in 373. For now let's use the idea to make some estimates. For example consider a quantum particle in one dimension, confined by impenetrable barriers at $x = 0$ and $x = L$. In this case we can estimate $\Delta x$ as $L$. In such a case there is also a lower bound to the particle's kinetic energy! It cannot have $K = 0$. Let's see why. In this one-dimensional case, $\langle p \rangle = 0$ since $p$ is equally likely to be toward $x = 0$ or $x = L$. Therefore $(\Delta p)^2 = \langle p^2 \rangle$. Then for a non-relativistic system, $\langle K \rangle = (\langle p^2 \rangle)/(2m)$ for a particle of mass $m$. The uncertainty relation thus gives $\Delta p \sim \hbar/(2L)$ which results in

$$\langle K \rangle \simeq \frac{\hbar^2}{8mL^2}.$$ 

The basic physical concept is that since the barriers reduce and limit the uncertainty in $x$, the uncer-
tainty in $p$ must correspondingly increase. The system has a lowest possible kinetic energy, even though the particle is otherwise free.

In quantum field theory, forces are described as due to field particles (called bosons) being exchanged between the interacting particles of matter (called fermions). Because it takes a finite time, at the speed of light, for such a boson to travel from one fermion to another, the boson and its energy can be created out of nothing. If the boson created has a mass, then the range of the force is severely limited. We can write $\Delta t = \hbar/(2\Delta E)$ and $r_{\text{max}} = c\Delta t$ so $r_{\text{max}} = (\hbar c)/(2mc^2) = \hbar/(2mc)$. Particles whose temporary existence is allowed by the uncertainty principles are called "virtual" particles. Note that when a very massive boson must be created, the force will have an incredibly short range... this is the case with the so-called "weak interaction," one of the four fundamental forces of nature. On the other hand when the mass of the field boson is zero, as is the case for the photon, the force has an infinite range.

Free particle in three dimensions: Just as an example, the state function for a free particle, with defi-
finite momentum $\mathbf{p}$ and energy $E$, looks like

$$\Psi_{fp}(\mathbf{r}, t) = A \exp[(i/\hbar)(\mathbf{p} \cdot \mathbf{r} - Et)].$$

Note the state is defined over all space, there is no position information at all for the particle. In physics jargon this is often called a "plane wave," but it is a sum of real and imaginary parts that are each wavelike but precisely out of phase. [Remember $\exp(iy) = \cos(y) + i\sin(y)$.] Also note that the probability distribution for this state is a constant... the particle has the same probability of being anywhere in space-time!