Quantum Introduction:

The fundamental description quantum physics provides is the state function $\psi(r, t)$. The measurable quantity is the probability distribution,

$$P(r, t) = |\psi(r, t)|^2.$$

For a state which has definite energy $E$ the probability distribution is time-independent, and we can leave off the $t$-dependence of the state function: $P(r) = |\psi(r)|^2$.

Let’s consider a system with just one degree of freedom, so that we only need to find $\psi(x)$.

Observables:

In quantum physics, observable quantities are represented by operators, which operate on $\psi$. The total energy operator is called $\hat{H}$, the kinetic energy operator $\hat{K}$, and so on.

Let’s try to figure out some of the operators. We will always consider a system with definite energy. In that case the operator for energy can be replaced by the energy itself.

For particles with definite magnitude of momentum, $\psi$ is wavelike with $\lambda = h/p$. Try $\psi(x) = A \sin(kx)$. 
Note \((d^2\psi/dx^2) = -k^2\psi\). But \(k = p/\hbar\), and non-relativistically \(K = p^2/(2m) = E - U(x)\).

Therefore \(k^2 = 2m(E - U)/\hbar^2\).

This leads to the one-dimensional Schrödinger Equation:

\[
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U(x) \right] \psi(x) = E\psi(x).
\]

Note that \(\hat{K} = -[\hbar^2/(2m)](d^2/dx^2)\), for example, and \(\hat{U} = U\).

Classic example: an impenetrable, one-dimensional box of length \(L\). In other words, assume \(U = 0\) for \(0 < x < L\), and there are impenetrable walls at \(x = 0\) and \(x = L\). Then we must have standing waves of probability between the walls. In other words, we can only have \(\lambda = 2L, L, L/2\), etc. We write this as \(\lambda_n = 2L/n\), where \(n\) is an integer equal to or greater than 1. \(n\) is called the principal quantum number. In general we have a quantum number for each degree of freedom of a system.

Thus \(k_n = 2\pi/\lambda_n = n\pi/L\).

\[\psi_n(x) = A\sin(n\pi x/L).\]

Plug into the Schrödinger equation and you will find
that this is a solution only if the total energy is

$$E_n = \frac{n^2 \hbar^2}{8mL^2}.$$  

It is customary to normalize states that are "bound," using

$$\int_{-\infty}^{\infty} \psi^2(x) dx = 1.$$  

The result is that

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left[n\pi x/L\right].$$
Quantum Harmonic Oscillator:
A particle in a potential $U(x) = (1/2)kx^2 = (1/2)m\omega^2x^2$, has energies:

$$E_n = \hbar\omega(n + 1/2).$$

Tunneling:
A quantum particle can enter a region where $K < U$, but in this region its state function is not wavelike, but behaves like $\exp(-2\alpha x)$ where $\alpha$ is proportional to $\sqrt{U - K}$.

Classic example: tunneling of a particle through a barrier.

Finite Potentials:
If a potential $U$ is finite, then the state functions can penetrate the region where $K < U$, although they fall off rapidly in that non-classical regime. Unless there is another similar potential nearby, the particle cannot “leak out” of the potential it starts in. But if there are similar potentials nearby, the particle can jump from potential to potential.
3D Box:

Three-dimensional systems have at least 3 quantum numbers. A classic example is an impenetrable cube of side length $L$. The usual standing wave argument would lead immediately to energies

$$E_{n_x,n_y,n_z} = (\frac{h^2}{8mL^2})[n_x^2 + n_y^2 + n_z^2],$$

where each principle quantum number is an integer 1 or more.

Degeneracy:

A system is said to be degenerate if two or more different sets of quantum numbers lead to the same energy. For example in the cube, the states $[n_x, n_y, n_z]$ with numbers $[1, 1, 2]$, $[1, 2, 1]$, and $[2, 1, 1]$ have the same energy. In quantum physics, degeneracy is generally due to a symmetry. For example, in the cube case, if you swap around the labels $x$, $y$, and $z$, nothing changes. This ability to swap coordinate labels without affecting physics is a symmetry of this system.

Spin:

All particles in nature have an intrinsic quantum number called "spin," $s$. The spin quantum number is easily observable because for charged particles
it is associated with a magnetic moment. The spin quantum number can only be an integer or a half integer. That is, it can only be 0, 1/2, 1, 3/2, 2, etc. It never changes.

Particles with an integer spin are called bosons. Particles with a half-integer spin are called fermions. There is a big difference. Bosons are not conserved in number and can be created out of nothing... classic example, the photon, which has spin 1. Fermions are conserved in number... classic example, the electron, which has spin 1/2. Fermions obey an exclusion principle... no two identical fermions can have all the same quantum numbers, in the same region of space. Bosons behave exactly the opposite. Any number of bosons can have precisely the same quantum numbers in the same region of space.

**Relativistic Quantum Physics**

When quantum physics was brought into line with special relativity, two important discoveries resulted. First, it was not possible to make the dynamical equations relativistic without including spin. Second, and most-earth-shaking, there were two solutions to the equations!

What this meant was that every particle has a part-
ner, an antiparticle, which is precisely like it except for the value of some intrinsic quantum number. For example, the electron had an antiparticle, the positron, which was precisely like it in all details except for having a positive charge. Similarly, the antiproton has a negative charge, whereas the proton has a positive charge.

The number of identical fermions can be considered a conserved quantum number. For example if we assign $L = 1$ to electrons, then positrons have $L = -1$. What this means is that particles and antiparticles which are fermions can only be created and destroyed in pairs.

A photon passing near a heavy nucleus can create an electron-positron pair. The photon kinetic energy converts into the total energy of the electron-positron system. Similarly, if an electron encounters a positron, they can annihilate into two photons. Again, the total energy of the electron-positron system is converted into the kinetic energy carried away by the two photons. These processes do not violate conservation of $L$, for example, because $L = 0$ before and after. Similarly, charge, momentum and all other such quantities are conserved.

By the way, what would be the antiparticle of the
photon? The answer is that the photon does not carry any intrinsic quantum numbers that could be different for an antiparticle, so the photon is its own antiparticle. This is not an unusual situation in fundamental physics.