SUPERCONDUCTING LC OSCILLATOR:

\[ L(dI/dt) + Q/C = 0. \]

or

\[ L(d^2Q/dt^2) + Q/C = 0. \]

Compare to

\[ m(d^2x/dt^2) + kx = 0. \]

We see that \( L \) is like \( m \) and \( C \) is like \( 1/k \). We know \( x(t) = x_{\text{max}} \cos \omega t \), where \( \omega = \sqrt{k/m} \).

Thus \( Q(t) = Q_{\text{max}} \cos(t/\sqrt{LC}) \). Use \( I = dQ/dt \) and we finally get

\[ I(t) = -I_{\text{max}} \sin(t/\sqrt{LC}). \]

Here \( I_{\text{max}} = Q_{\text{max}}/\sqrt{LC} \).

Also \( U = Q^2/(2C) + (1/2)LI^2 \) and plugging in gives

\[ U = Q_{\text{max}}^2/(2C) \]

as expected.
**LRC CIRCUIT:**

\[ L\left(\frac{dI}{dt}\right) + IR + \frac{Q}{C} = 0. \]

The three possible solutions are called (1) overdamped, (2) critically damped, and (3) underdamped.

The underdamped solution looks like:

\[ Q = Q_{\text{max}} \exp(-\beta t) \cos \omega_d t, \]

where \( \beta = R/2L \) and \( \omega_d = \sqrt{\omega_0^2 - R^2/(4L^2)} \), with \( \omega_0 \) the natural frequency of the undamped oscillator.

Damped oscillators are usually classified as to their \( Q \)-value, defined by

\[ Q = -2\pi(U/\Delta U). \]

For a damped oscillator

\[ U = [Q_{\text{max}}^2/(2C')] \exp(-Rt/L), \]

and since \( \Delta U = (dU/dt)\Delta t = -(UR/L)\Delta t \), if we let \( \Delta t = T \simeq 2\pi/\omega_0 \), we finally get

\[ Q = \omega_0L/R. \]

One way to understand the physical significance of \( Q \) is to note the number of oscillations in a time \( t = 2L/R \) is

\[ N = t/T = Q/\pi. \]
SIDELIGHTS ON AC CIRCUITS:

If an AC circuit consists only of a resistance, and \( \Delta V(t) = \Delta V_{\text{max}} \sin \omega t \), it is obvious from Ohm’s Rule that the current \( I(t) \) tracks the voltage precisely.

\[
I(t) = \frac{\Delta V_{\text{max}}}{R} \sin \omega t,
\]

and \( I_{\text{max}} = \Delta V_{\text{max}} / R \).

If a circuit also has a pure inductance \( L \) and a pure capacitance \( C \), then no power can be lost in either the inductor or capacitor. All power is lost in the resistor.

\[
\mathcal{P} = I^2 R = I_{\text{max}}^2 R \sin^2 \omega t.
\]

We get the average power over a cycle of voltage oscillation by

\[
\mathcal{P}_{\text{avg}} = (1/T) \int_{0}^{T} I_{\text{max}}^2 R \sin^2 \omega t \, dt.
\]

Let \( x = \omega t \) so \( dx = \omega dt \). Then the integral becomes

\[
\mathcal{P}_{\text{avg}} = \frac{I_{\text{max}}^2 R}{\omega T} \int_{0}^{\omega T} \sin^2 x \, dx.
\]

But of course \( \omega T = 2\pi \). So the final result of the integration is

\[
\mathcal{P}_{\text{avg}} = \frac{1}{2} I_{\text{max}}^2 R.
\]
We define \textit{root mean square} values by

\[ A_{\text{rms}} = A_{\text{max}} / \sqrt{2}. \]

The rms voltage and current will be the basis of our further analysis.

\[ P_{\text{avg}} = I_{\text{rms}}^2 R. \]

This approach makes AC relations look as much as possible like the DC relations we have seen so frequently.
**Pure Inductive AC Circuit**

Using the Kirchhoff Loop rule, $\Delta V - LdI/dt = 0$.

Thus $LdI/dt = \Delta V_{\text{max}} \sin \omega t$.

This means $dI = \Delta V_{\text{max}} / L \sin \omega t dt$.

Thus we take

$$I_L = -\frac{\Delta V_{\text{max}}}{\omega L} \cos \omega t.$$  

For comparison with the resistor result we can write

$$-\cos \omega t \text{ as } \sin(\omega t - \pi/2).$$

It is usually said that in this case the current “lags” the voltage by a phase angle of 90°. It makes more sense to say that the inductor voltage “leads” the current by such an angle, and therefore would “lead” any resistor voltage by the same angle.

Note that for a pure-inductor circuit,

$$I_{\text{max}} = \Delta V_{\text{max}} / (\omega L).$$

Heaviside defined the “inductive reactance,” in analogy with resistance, by $X_L = \omega L$. Then $I_{\text{max}} = \Delta V_{\text{max}} / X_L$.

Note that therefore $I_{\text{max}} \to \infty$ as $\omega \to 0$. 
PURE CAPACITOR AC CIRCUIT:

Now $\Delta V = \Delta V_C = \Delta V_{\text{max}} \sin \omega t$. Since $Q = CV$, we must have $Q(t) = C\Delta V_{\text{max}} \sin \omega t$ and since $I = dQ/dt$,

$$I_C(t) = C\Delta V_{\text{max}} \omega \cos \omega t.$$ 

To compare with the normal voltage, we use $\cos \omega t = \sin(\omega t + \pi / 2)$. Then

$$I_C(t) = C\omega \Delta V_{\text{max}} \sin(\omega t + \pi / 2).$$

It is sometimes said that the capacitor voltage "lags" the current, and therefore $\Delta V_R$, by $90^\circ$.

Notice that $I_{\text{max}} = \omega C \Delta V_{\text{max}}$, so Heaviside defined the capacitative reactance $X_C = 1/(\omega C)$ so that $I_{\text{max}} = \Delta V_{\text{max}} / X_C$, an AC version of Ohm's rule for capacitance.

For a purely inductive circuit, the current and $\Delta V_R$ lag the inductor voltage by $\pi / 2$. For a purely capacitative circuit, the current and $\Delta V_R$ lead the capacitor voltage by $\pi / 2$. This leads to a famous phasor diagram!
PHASORS:

For a series AC circuit,

\[ \Delta V(t) = \Delta V_{\text{max}} \sin \omega t, \]

\[ I(t) = I_{\text{max}} \sin(\omega t + \phi), \]

\[ \Delta V_R(t) = I_{\text{max}} R \sin \omega t, \]

\[ \Delta V_L(t) = I_{\text{max}} X_L \sin(\omega t + \pi/2), \]

and \( \Delta V_C(t) = I_{\text{max}} X_C \sin(\omega t - \pi/2), \)

where \( X_L = \omega L \) and \( X_C = (\omega C)^{-1} \).

When you draw the “phasors” it becomes obvious that \( \tan \phi = (X_L - X_C)/R \) and

\[ \Delta V_{\text{max}} = \sqrt{\Delta V_R^2 + (\Delta V_L - \Delta V_C)^2}, \]

which is

\[ I_{\text{max}} \sqrt{R^2 + (X_L - X_C)^2}, \]

so that

\[ I_{\text{max}} = \frac{\Delta V_{\text{max}}}{Z} \]

where

\[ Z = \sqrt{R^2 + (X_L - X_C)^2}, \text{ and } \tan \phi = \frac{X_L - X_C}{R}. \]
**Power**

Using $P = I \Delta V$ it is easy to show that, averaged over a cycle,

$$P_{\text{avg}} = \frac{1}{2} I_{\text{max}} \Delta V_{\text{max}} \cos \phi = I_{\text{rms}} \Delta V_{\text{rms}} \cos \phi,$$

and therefore that

$$P_{\text{avg}} = I_{\text{rms}}^2 R.$$

**Resonance:**

$I_{\text{rms}}$ is a maximum when $X_L = X_C$ or $\omega_0 = 1/\sqrt{LC}$. This leads to

$$P_{\text{avg}} = \frac{\Delta V_{\text{rms}}^2 R \omega^2}{R^2 \omega^2 + L^2 (\omega^2 - \omega_0^2)^2}.$$

**Transformer:**

Using Faraday’s Law we can instantly see that

$$\frac{\Delta V_1}{N_1} = \frac{\Delta V_2}{N_2} \text{ and } I_1 \Delta V_1 = I_2 \Delta V_2.$$
Remember that in empty space

\[ \oint \mathbf{E} \cdot d\mathbf{l} = -(d/dt) \oint \mathbf{B} \cdot d\mathbf{A}, \]

and Maxwell realized there is a corresponding equation that looks like

\[ \oint \mathbf{B} \cdot d\mathbf{l} = +\mu_0\varepsilon_0 (d/dt) \oint \mathbf{E} \cdot d\mathbf{A}. \]

In other words, he realized a changing electric field in empty space can generate a magnetic field in empty space.

We can combine Maxwell’s Law with Ampere’s Law, to get

\[ \oint B \cdot d\mathbf{l} = \mu_0 I_{\text{inc}} + \mu_0\varepsilon_0 (d/dt) \oint \mathbf{E} \cdot d\mathbf{A}. \]
Between circular capacitor plates, the electric field varies according to $\mathbf{E}(t) = \mathbf{E}_0[1 - t/\tau]$, for $0 < t < \tau$. What magnetic field $\mathbf{B}(r)$ is created between the plates during that interval? Here $r < R$, the radius of the plates.

$$\oint \mathbf{B} \cdot d\bar{l} = \mu_0 \varepsilon_0 (d/dt) \int \mathbf{E} \cdot dA.$$ 

Using symmetry we get $2\pi r B = \mu_0 \varepsilon_0 E_0 (-1/\tau) \pi r^2$, so the final result is

$$B(r) = -\frac{\mu_0 \varepsilon_0 E_0 r}{2\tau}.$$ 

The minus sign means that $\mathbf{B}$ is in the opposite direction to $d\bar{l}$. 
In an AC circuit we have an angular frequency of 1 rad/s, a resistance of 1 Ohm, an inductance of 2 H, a capacitance of 1 F and $\mathcal{E}_{\text{max}}$ of 10 V. Find $f$, $Z$, $I_{\text{max}}$, $\phi$, $\Delta V_R$, $\Delta V_L$ and $\Delta V_C$.

\[ f = \frac{\omega}{2\pi} = 0.159 \text{ Hz}. \]
\[ X_L = \omega L = 2 \Omega. \]
\[ X_C = \frac{1}{(\omega C)} = 1 \Omega. \]
\[ Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{2} \Omega. \]
\[ I_{\text{max}} = \frac{\Delta V_{\text{max}}}{Z} = 7.07 \text{ Amps}. \]
\[ \phi = \tan^{-1}\left[\frac{X_L - X_C}{R}\right] = \tan^{-1}(1) = 0.785 \text{ rad or } 45^\circ. \]

Thus $\Delta V_R = \Delta V_C = 7.07 \text{ V}$, and $\Delta V_L = 14.1 \text{ V}$. 
Suppose an LRC circuit has $\mathcal{E}(t) = \mathcal{E}_{\text{max}} \sin \omega t$, with frequency $50/\pi$ Hz, $\mathcal{E}_{\text{max}}$ of 40 V, L of 160 mH, C of 99$\mu$F, R of 68 Ohms. What is $I_{\text{max}}$, what is $\phi$, what is $I(t)$? What is the earliest time for which $I = 0$? What average power is the AC source delivering? What is the resonant angular frequency $\omega_R$? What is $I_{\text{max}}$ at resonance?

We can easily calculate $\omega = 100 \text{ rad/s}$, $X_L = 16 \Omega$, $X_C = 101 \Omega$, $Z = 109 \Omega$.

Then $I_{\text{max}} = 0.37 \text{ Amps}$, $\phi = -0.896 \text{ rad}$, and

$$I(t) = [0.37 \text{ A}] \sin[(100 \text{ s}^{-1}t) - 0.896].$$

The average power delivered can be calculated two different ways as a check:

$$\mathcal{P}_{\text{avg}} = I_{\text{max}}^2 R / 2 = 4.65 \text{ Watts}.$$  

$$\mathcal{P}_{\text{avg}} = (1/2) I_{\text{max}} \Delta V_{\text{max}} \cos \phi = 4.6 \text{ Watts}.$$  

The resonant frequency is $\omega = 1/\sqrt{LC} = 251 \text{ rad/s}$ or 40 Hz.

The maximum current at the resonant frequency is $\Delta V_{\text{max}}/R = 0.59 \text{ Amps}$.