Faraday's Law:

\[ \mathcal{E} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A}. \]

Lenz's Law:

The explanation of the minus sign in Faraday's Law: Induced currents, emfs and fields are always opposed to the change in \( \int \mathbf{B} \cdot d\mathbf{A} \). For example, if a magnetic field in a certain direction is getting larger, the induced magnetic field will have to point in the opposite direction, to cancel the change. If the conducting loop is getting smaller, the induced field would have to add to the original field, again to oppose the change.
Right Hand Rule for Surfaces:

When we set up induction calculations using Faraday's law or similar laws involving magnetic flux through a surface, we can build in the right-hand rule by using a right-hand rule to connect the differential area $d\mathbf{A}$ of a point on the surface to the differential $d\mathbf{l}$ of the path we take around the surface area element. The general idea is that if we take $d\mathbf{A}$ in a certain direction, then $d\mathbf{l}$ is tangent to the path taken around the boundaries of the area, in a right-hand sense.
**Lenz's Law:**

Induced currents, emf's and fields always oppose the system changes that create them.

\[ \vec{B} \text{ increasing} \]

\[ \int \vec{B} \cdot d\vec{A} > 0 \quad I < 0 \] (opposite to \( d\vec{A} \))

\[ \vec{B} \text{ weakening!} \]

So that...
Faraday’s Law of Induction:

\[ \mathcal{E} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A}. \]

Lenz Law:
Induced currents and induced fields always oppose the flux change that induced them.

**Induced E Fields:**

\[ \oint \mathbf{E} \cdot d\mathbf{l} = \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A}. \]

**Inductance:**

\[ \mathcal{E}_2 = -M \frac{dI_1}{dt} \text{ and } \mathcal{E}_1 = -M \frac{dI_2}{dt}. \]

**Self Inductance:**

\[ \mathcal{E} = -L \frac{dI}{dt}. \]

**Energy in Inductor:**

\[ U = \frac{1}{2} LI^2. \]
Circuit with Resistor and Inductor:

\[ I(t) = \frac{E}{R} \left[ 1 - \exp(-tR/L) \right]. \]

Energy Density of Magnetic Field:

\[ u = \frac{B^2}{2\mu_0}. \]
• A coil of fixed size with 1000 turns, total resistance 100 Ohms, and radius $a = 0.1 \text{ m}$, is in a region where a uniform magnetic field increases from 0.00 to 0.01 T in 0.001 sec. What is the average induced current during this interval?

$$\mathcal{E} = -(d/dt) \int \mathbf{B} \cdot d\mathbf{A}.$$ This becomes, for this case, $$\mathcal{E} = -NA(\Delta B/\Delta t),$$ where $N$ is the number of turns in the coil, and $A = \pi a^2$ is the area of each turn. Thus $$\mathcal{E} = -NA(B_f - B_i)/\Delta t$$ and $$I = |\mathcal{E}|/R$$ so finally

$$I = (N\pi a^2/R)(B_f - B_i)/\Delta t,$$

and plugging in the numbers gives $I = \pi$ Amps. Since $\mathcal{E}$ is negative, this means that the induced current is in the opposite direction to $d\mathbf{\ell}$ when it is defined to be consistent with $\mathbf{A}$ pointing in the direction of $\mathbf{B}$, as we assumed in the setup.

• A loop of wire is pulled through a plastic tube, which changes the area of the loop, its radius changing from 0.1 m to 0.01 m in 0.1 sec. A uniform magnetic field was passing through the loop. If we take $d\mathbf{\ell}$ and $d\mathbf{A}$ as shown in the sketch, so that $d\mathbf{A}$ is in the same direction as $\mathbf{B}$, we get

$$\mathcal{E} = -(d/dt) \int \mathbf{B} \cdot d\mathbf{A} = -B(\Delta A/\Delta t)$$
which is $\mathcal{E} = -\pi B(r_f^2 - r_i^2)/\Delta t$. Now suppose the loop has resistance 10 Ohms, and the constant magnetic field strength is $10^{-4}$ T. What is the current during the interval in which the loop is decreased by pulling?

The current is $I = |\mathcal{E}|/R = (\pi B/R\Delta t)(r_f^2 - r_i^2)$ and plugging in the numbers gives $3 \times 10^{-6}$ Amps. Note that since $r_f < r_i$, the induced emf is positive from Faraday’s law. Therefore the induced current is in the same direction as $d\vec{l}$. The induced magnetic field is in the same direction as the uniform field.

- A loop of $N$ turns is rotating in a uniform magnetic field. The angle between $\mathbf{B}$ and $d\mathbf{A}$ is $\theta = \omega t$. Suppose the magnitude of $\mathbf{B}$ is 0.6 T, the radius of the loop is 0.3 m, the number of turns is 10, and $\omega = \pi/2$ per sec. What is $\mathcal{E}$ during the time interval from 0.0 to 1.0 sec?

Again we apply Faraday’s law and get

$$\mathcal{E} = +BNA\omega \sin(\omega t).$$

Plugging in the numbers gives $\mathcal{E}(t) = 2.66 \, V \times \sin(\pi t/2)$. Note that the overall + sign means that the induced current is in the same direction as $d\vec{l}$. 

Maxwell pointed out that Faraday’s law can be written in empty space as $\oint \mathbf{E} \cdot d\mathbf{l} = -(d/dt) \int \mathbf{B} \cdot d\mathbf{A}$. Suppose in empty space between the cylindrical pole faces of an electromagnet, $dB/dt$ is given. Suppose the pole faces have a radius $R$. What electric field exists between the pole faces for $r < R$, and for $r > R$?

Consider a loop of radius $r$. Then the $\mathbf{E}$ integral gives $2\pi r E$. The other integral becomes $\pi r^2 dB/dt$ so the final result is

$$E(r < R) = -(r/2)dB/dt.$$  

The minus sign means that $\mathbf{E}$ points in the opposite direction to $d\mathbf{l}$ as defined by $d\mathbf{A}$, if $dB/dt > 0$. Outside the pole faces, $E(r > R) = -(R^2/(2r))dB/dt$.

If we define the self-inductance $L$ by $\mathcal{E} = -L(dI/dt)$, show that the energy stored in an inductor when the current is brought up from 0 to $I$ is $U = (1/2)L I^2$.

The power being delivered at a given $I$ is $P = I\mathcal{E} = -LI(dI/dt)$. By definition of power, $dU = P dt = LI(dI/dt) dt = LI dI$. Integrating from 0 to $I$ gives $U = (1/2)L I^2$. This should be compared to similar relations for a capacitor.