Suppose we have the usual free-particle state, satisfying

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle, \text{ where } H = (1/2m)\mathbf{p}^2.$$

Obviously if we make a local gauge transformation, $|\psi'\rangle = \exp iQ\epsilon(\mathbf{r},t)|\psi\rangle$, we get derivatives of the arbitrary function $\epsilon(\mathbf{r},t)$ and so $|\psi'\rangle$ does not satisfy the Schrödinger equation.

However, suppose the system were coupled to a *vector field which is gauge invariant!*

$$H = (1/2m)(\mathbf{p} - (q/c)\mathbf{A})^2 + qA_0.$$

The gauge freedom provided by the $(A_0, \mathbf{A})$ field allows us to cancel the unwanted terms and recover the same equation, satisfied by *both* $|\psi'\rangle$ and $|\psi\rangle$.

$$A_0' = A_0 - \hbar\frac{\partial\epsilon}{\partial t}, \ \mathbf{A}' = \mathbf{A} + \hbar c\nabla\epsilon.$$

Note we could define $A'^{\mu} = A^{\mu} - \hbar c\nabla^{\mu}\epsilon$, where

$$\nabla^{\mu} = (\frac{1}{c}\frac{\partial}{\partial t}, -\nabla).$$

We can even include the gauge field automatically by defining a covariant derivative, like $D_\mu = (D_0, \mathbf{D})$, where

$$D_0 = (1/c)\frac{\partial}{\partial t} + \frac{iqA_0}{\hbar c},$$

and

$$\mathbf{D} = \nabla - \frac{iq\mathbf{A}}{\hbar c}.$$

To stress the point: *requiring* local gauge invariance automatically generates a gauge-invariant field, and thus automatically includes interactions with the field, such as qA_0 and $\mathbf{j} \cdot \mathbf{A}$.

According to Maxwell's Equations

$$\frac{1}{c^2}\frac{\partial^2 A_0}{\partial t^2} - \nabla^2 A_0 = \rho_q = \langle\psi|q|\psi\rangle.$$

$$\frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} - \nabla^2 A_i = \frac{j_i}{c} = \langle\psi|qv_i/c|\psi\rangle.$$

Gauge freedom allows us to choose, for example, the Lorentz Gauge:

$$\frac{1}{c}\frac{\partial A_0}{\partial t} + \nabla \cdot \mathbf{A} = 0.$$

A simple covariant form is obtained as $\Box A^\mu = j^\mu/c$, where $\Box = \nabla_\mu \nabla^\mu$.

For a massive boson we would have an equation looking like the Klein-Gordon equation,

$$\Box A^\mu - \frac{(m_\gamma c)^2}{\hbar^2} A^\mu = \frac{j^\mu}{c}.$$

The mass term destroys gauge invariance! So at first it looked as if it would be impossible to have a gauge theory of weak interactions.

Aharonov-Bohm Effect

Imagine a very long solenoid, where $\mathbf{B}$ is uniform inside and basically zero outside. In this case the vector potential $\mathbf{A}$ circles the solenoid outside, forming closed loops. That is $\mathbf{B} = \nabla \times \mathbf{A} = 0$, but $\mathbf{A}$ is not zero.

Write the Schrödinger equation for the outside of the solenoid:

$$\Big[(1/2m)\big(\frac{\hbar\nabla}{i} - q\mathbf{A}/c)^2 + V\Big]\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

Make a transformation $\psi = \exp(ig)\psi'$, where

$$g(\mathbf{r}) = (q/\hbar c)\int_x^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}',$$

where we are integrating along a path.

Now look at $\nabla\psi$ and use the fact that $\nabla g = q\mathbf{A}/c$. The result (see text) is that

$$(\frac{\hbar\nabla}{i} - q\mathbf{A}/c)^2\psi = -\hbar^2 \exp(ig)\nabla^2\psi',$$

so if we cancel $\exp ig$ in every term we get

$$\frac{-\hbar^2}{2m}\nabla^2\psi' + V\psi' = i\hbar\frac{\partial\psi'}{\partial t}.$$

What this means is that any solution for no vector field $\mathbf{A}$ can be converted into a solution for a vector field with $\nabla \times \mathbf{A} = 0$ simply by multiplying the free particle solution by

$$\exp ig = \exp[(iq/\hbar c)\int \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r}].$$

Therefore if we integrate along a circular path around the solenoid, everywhere tangent to $\mathbf{A}$, and we have split a beam of particles into two beams that go around the opposite halves of the circular path, on one side $d\mathbf{r}$ is parallel to $\mathbf{A}$, and on the other side it is antiparallel, so if we define Φ as the usual magnetic flux through our loop, we get $g = \pm(q\Phi)/(2\hbar c)$

and the total phase difference between the two paths is $(q\Phi)/(\hbar c)$. This is easily detected by merging the two beams again and examining their interference with one another.

Yang-Mills Fields:

Suppose we had a gauge-invariant vector field consisting of two or more noncommuting massless vector fields. There would need to be some additive quantum number to distinguish the bosons, and the inevitable result would be that the fields couple to one another! These are called non-Abelian fields, and the real problem is that in such a case, free fields do not exist, because the fields themselves interact. The work of Yang and Mills on such fields had been largely forgotten when people started to work on a gauge theory of the strong interaction. The bosons of this field are massless, but they carry the source of the field, color. So the strong field is very precisely a Yang-Mills field!

Higgs Mechanism:

Recall $\Box A^\mu = j^\mu$. This looks like the KG equation but with a source term. If we remove the source, the obvious solution is no field, $A^\mu = 0$. [Of course the electromagnetic field has a radiation component, which seems to propagate indefinitely through empty space, but it had a source, accelerating charge, at some point in space.]

Now in general consider some field that satisfies $\Box\psi = -K\psi$. If $K = (mc/\hbar)^2$, this is the KG equation for a spinless particle with mass m, and there is no source term.

Now let's suppose we have two fields, ψ_1, ψ_2. And suppose they interact. Then we could, for example, work in terms of two coupled equations that look like

$$\Box\psi_1 = -[K_1 + k_{12}\psi_2^2]\psi_1, \ \Box\psi_2 = -[K_2 + k_{21}\psi_1^2]\psi_2.$$

In general for many interacting fields, we would have

$$\Box\psi_i = -[K_i + \sum_j k_{ij}\psi_j^2]\psi_i.$$

An important thing to notice is that the interactions are contributing to what could be a mass factor.

Step back to just one field, ψ_1, no interactions. If $K_1 = 0$ the obvious solution is $\psi_1 = 0$. Even if we have $K_1 > 0$, there is no source term, so the simplest solution is still $\psi_1 = 0$.

But suppose the field can interact *with itself*. Then we could get an equation that looks like

$$\Box\psi_1 = -[K_1 + k_{11}\psi_1^2]\psi_1.$$

There are now three solutions that are consistent with $\Box\psi_1 = 0$. We could have $\psi_1 = 0$ to begin with, OR we could have $K_1 + k_{11}\psi_1^2 = 0$.

Thus we get two NON-ZERO solutions,

$$\psi_1 = \pm\sqrt{-K_1/k_{11}}.$$

Now remember the usual way to deduce a function ϕ for the potential energy, given information like this.

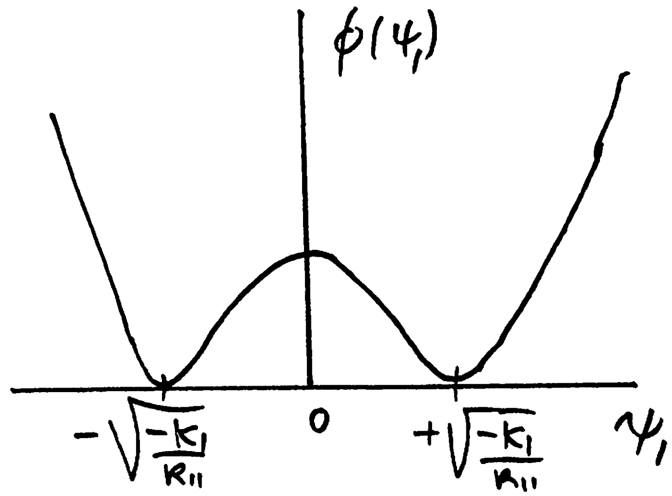
$$-\frac{\partial\phi}{\partial\psi_1} = -[K_1 + k_{11}\psi_1^2]\psi_1.$$

The result is

$$\phi(\psi_1) = \frac{1}{2}K_1\psi_1^2 + \frac{1}{4}k_{11}\psi_1^4.$$

To break symmetry, suppose $K_1 < 0$. Now there are 3 extrema, in the famous "Mexican Hat Potential."

Since ψ_1 is in general complex, what we have drawn is just a cut across the actual three-dimensional function.



We only find two stable solutions. The fact that this self-coupled field has two excitations means that if we couple it to other fields, for example

$$\Box\psi_j = -[K_j - k_{j1}(K_1/k_{11})]\psi_j,$$

then even when $K_j = 0$, the field j has a mass term.

$$\Box\psi_j = k_{j1}[K_1/k_{11}]\psi_j, \; K_1 < 0.$$

Therefore if we compare to the usual KG equation,

$$m_j = (\hbar/c)\sqrt{k_{j1}[-K_1/k_{11}]}.$$

The Standard Model of particle physics describes all pointlike fundamental particles, whether leptons, quarks or field bosons, as massless in the sense that $K_j = k_{jj} = 0$. These particles gain a mass by coupling to a unique, self-coupling field, the Higgs field. This field is governed by an equation with $K_1 < 0$ and $k_{11} > 0$. It thus has an effective mass even in its sourceless, steady state. Does the Higgs boson get its own mass from the Higgs field? Not necessarily, because the Higgs field itself could have a non-zero K_H. Also there may well be more than one Higgs field. Some of the simplest ways to go beyond the standard model have up to 5 Higgs fields!

It is important to stress again that the majority of the mass in the universe does not come from the Higgs field interactions. The masses of baryons and mesons do NOT come mainly from the Higgs interaction. Consider a particle made of constituents i. If this particle is in free space, and not interacting with external fields, classical relativity tells us

$$Mc^2 = \sum_i K_i + \sum_{ij} V_{ij} + \sum_i m_i c^2.$$

In the case of baryons and mesons most of the mass comes from the interaction terms, in other words, from the fields that bind the system together.

The Higgs field is the only example of a quantum scalar field known in nature, and thus its boson is the only example of a fundamental particle that is a 0^+ boson.

Spontaneous Symmetry Breaking and Goldstone Bosons:

Goldstone's theorem indicates that if a continuous symmetry exists but is spontaneously broken in actual physical systems, then a massless scalar particle (Nambu-Goldstone boson) appears for each generator of the broken symmetry.

In a system with gauge symmetry, whose field bosons are massless 1^- bosons with two polarization states, the symmetry breaking due to interaction with the Higgs field results in the Nambu-Goldstone bosons being eaten by the gauge bosons as they gain mass, to provide the extra spin substate, now $m_s = \pm 1, 0$, of the massive vector particles.

The "Mexican Hat" potential was originally suggested by Goldstone to illustrate this. Such a potential has an infinite number of vacuum states $\phi = A \exp(i\theta)$, for all real θ from zero to 2π. Once the system occupies a particular state, a specific θ, we can no longer see the original symmetry, but all values of θ correspond to the same energy. The massless mode is the circular mode that stays at the bottom of the trough. Excitations of the field correspond to oscillations back-and-forth across the circular path.