Table 1. Transformation properties of electric field $\boldsymbol{E}$, magnetic field $\boldsymbol{B}$, charge q, velocity $\boldsymbol{v}$ and force $\boldsymbol{F}$ under charge conjugation (C), parity (P) and time reversal (T).

	C	P	T
$\boldsymbol{E}$	$-\boldsymbol{E}$	$-\boldsymbol{E}$	$\boldsymbol{E}$
$\boldsymbol{B}$	$-\boldsymbol{B}$	$\boldsymbol{B}$	$-\boldsymbol{B}$
q	$-q$	q	q
$\boldsymbol{v}$	$\boldsymbol{v}$	$-\boldsymbol{v}$	$-\boldsymbol{v}$
$\boldsymbol{F}$	$\boldsymbol{F}$	$-\boldsymbol{F}$	$\boldsymbol{F}$

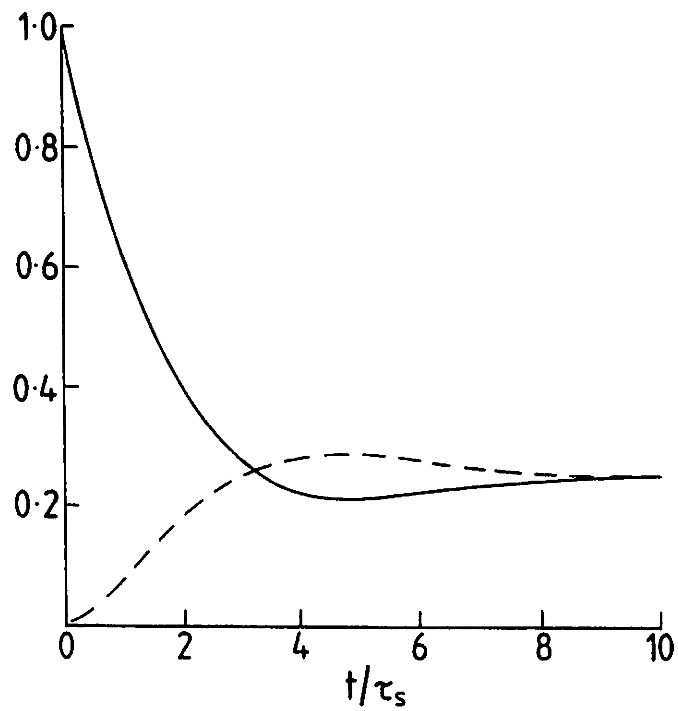


Fig. 9.13 Predicted variation with time of the intensities $I(K^0)$ (solid line) and $I(\bar{K}^0)$ (dashed line) for an initial K^0 beam. The curves are calculated using (9.46) for $\Delta m \cdot \tau_S = 0.5$, where Δm is the mass difference (9.47) and τ_S is the K-short lifetime.

Electromagnetism

$$\mathbf{B} = \nabla \times \mathbf{A}, \ \mathbf{E} = -\nabla A_0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}.$$

The "minimal electromagnetic substitution" is $H \rightarrow H - qA_0, \ \mathbf{p} \rightarrow \mathbf{p} - q\mathbf{A}/c$.

Therefore for a charged particle in free space, in an EM field,

$$H = \frac{1}{2m}[\mathbf{p} - q\mathbf{A}/c]^2 + qA_0 \simeq \frac{p^2}{2m} + H_{\text{int}},$$

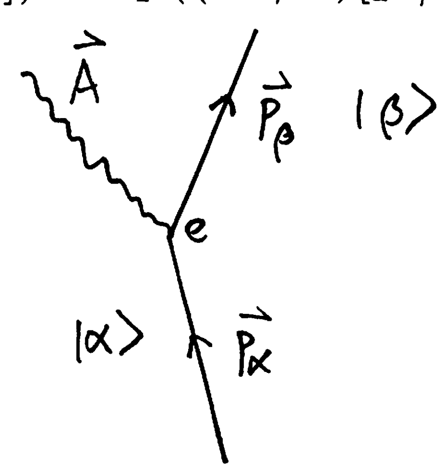
where to first order $H_{\text{int}} = -(q/mc)\mathbf{p} \cdot \mathbf{A} + qA_0$.

Therefore we describe a fundamental EM process by $H_{em} = (e/mc)\mathbf{p} \cdot \mathbf{A}$,

and with proper normalization, the photon state function can be taken as

$$\mathbf{A}\gamma = \mathbf{a}[\exp((i/\hbar)[\mathbf{p}_\gamma \cdot \mathbf{r} - E_\gamma t]) + \exp((-i/\hbar)[\mathbf{p}_\gamma \cdot \mathbf{r} - E_\gamma t])].$$

Here $\mathbf{a} = \hat{\boldsymbol{\epsilon}}\sqrt{\frac{2\pi\hbar^2 c^3}{E_\gamma V}}$.



So we are interested in $\langle\beta|H_{\rm em}|\alpha\rangle = \int d^3\mathbf{r}\psi_\beta^* H_{\rm em}\psi_\alpha$.

We take $H_{\rm em} = (e/mc)[-i\hbar\nabla]\cdot\mathbf{A}$.

In the simplest possible case we take the initial and final momenta of the particle to be negligible, and $E_\gamma = p_\gamma c$ small compared to $\hbar c/R$, and in that case the matrix element is like a constant times

$$\hat{\boldsymbol{\epsilon}}\cdot\int d^3\mathbf{r}\psi_\beta^*\nabla\psi_\alpha,\quad E_\beta = E_\alpha - E_\gamma.$$

If we now turn to "Fermi's Golden Rule of Time-Dependent Perturbation Theory," a general and useful expression for the transition probability per unit time,

$$dW_{\beta\alpha} = (2\pi/h)|\langle\beta|H_{\rm em}|\alpha\rangle|^2\rho(E_\gamma),$$

where the density of final states is

$$\rho(E_\gamma) = (E_\gamma^2 V d\Omega)/(2\pi\hbar c)^3,$$

then the result is

$$\frac{dW_{\beta\alpha}}{d\Omega} = \frac{e^2E_\gamma}{2\pi m^2c^3}|\hat{\boldsymbol{\epsilon}}\cdot\int d^3\mathbf{r}\psi_\beta^*\nabla\psi_\alpha|^2.$$

We can use a simple trick to get rid of the ∇, since the states are free-particle states. In fact we can replace the ∇ in the integral by $[mE_\gamma/\hbar^2]\mathbf{r}$.

The final result is.

$$\frac{dW_{\beta\alpha}}{d\Omega} = \frac{e^2}{2\pi\hbar^4 c^3} E_\gamma^3 |\hat{\epsilon} \cdot \langle\beta|\mathbf{r}|\alpha\rangle|^2.$$

Since the Poynting Vector is parallel to the photon momentum, we take $\hat{\epsilon}$ parallel to **E**. In spherical polar coordinates, take $\mathbf{p}_\gamma$ along z, and $\hat{\epsilon}$ in the x, y plane. With $\langle \mathbf{r} \rangle$ in the x, z plane and noticing there are two possible polarization states to sum over, we can do the angle integrals to obtain

$$W_{\alpha\beta} = \frac{4e^2 E_\gamma^3}{3\hbar^4 c^3} |\langle \beta | \mathbf{r} | \alpha \rangle|^2 = (1/\tau).$$

[See text for full details, including the various higher order terms that have been neglected here.]

Depending upon where in the charge distribution the photon pops into existence, it can carry ℓ as well as spin. More generally, the photon can be emitted by either the scalar charge distribution, or the charged probability current of the system, in other words, the source can be a polar vector or an axial vector. The fastest transition is the lowest order, the "dipole transition," if that's possible in the system.

As a result we distinguish two different kinds of electromagnetic transitions, so-called E and M. If j is the total photon angular momentum quantum number, the Ej transitions involve photon parity $\eta_P = (-1)^j$ and the Mj transitions $-(-1)^j$.

Since $\mathbf{J}_\alpha = \mathbf{J}_\beta + \mathbf{J}$, and $\eta_P(\alpha) = \eta_P(\beta)\eta_P$, clearly $|J_\alpha - J_\beta| \leq j \leq J_\alpha + J_\beta$.

Note that E1 involves a partity change, but M1 does not. E2 involves no parity change, but M2 does.

A full multipole expansion is discussed in many books. We will not present the gory details here. The important thing to realize is that transition rates for higher order transitions are tremendously suppressed compared to the lower orders.

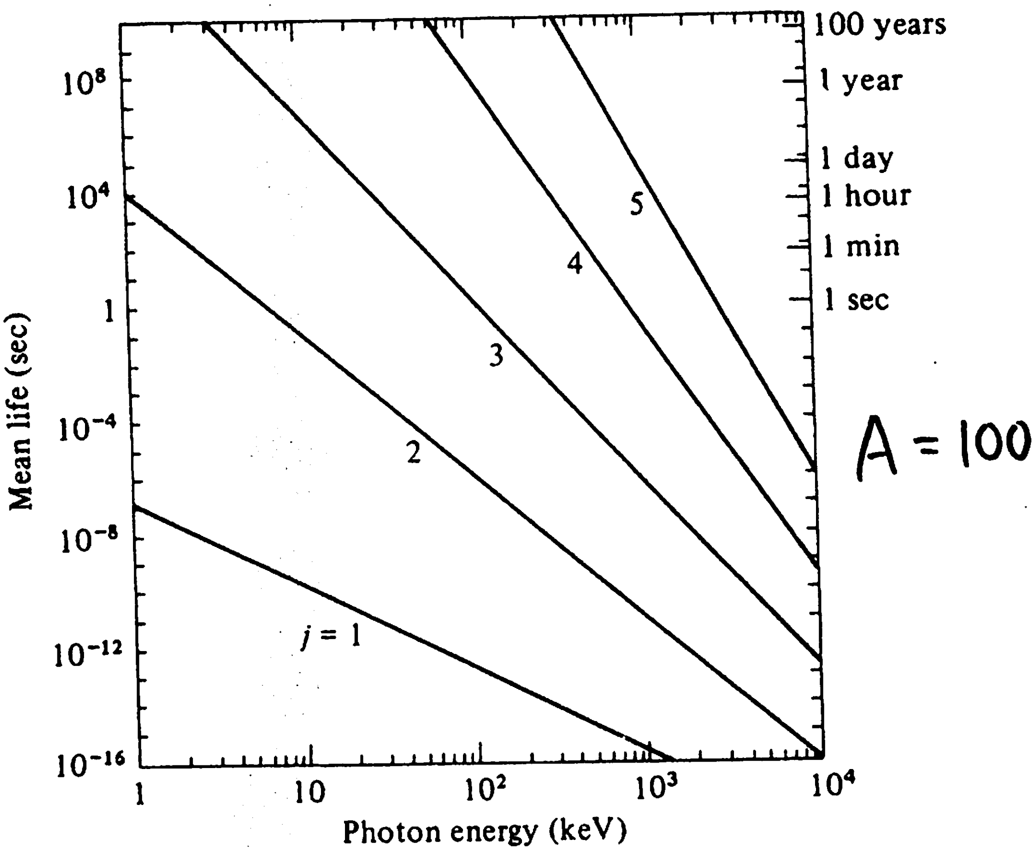


Fig. 7.6 Estimated mean lives for electric multi-pole radiation of order 2^j as a function of the energy of the emitted photon, for a nucleus with $A = 100$. Corresponding estimates for other nuclei may be obtained by multiplying by $(100/A)^{2j/3}$. Mean lives for magnetic multi-pole radiation are generally longer than those for electric multi-pole radiation of the same order by a factor $\tau_M/\tau_E \sim 20A^{\frac{2}{3}}$.

(The lines are drawn from formulae given, for example, in Jackson, J. D. (1975). *Classical Electrodynamics*, 2nd ed., New York: Wiley, p. 760.)

The atomic nucleus provides a fine example of elecromagnetic transitions. The charge radius for a nucleus with nucleon number A is $R \simeq 1.2\ \mathrm{fm} A^{1/3}$, so that $W_{\beta\alpha}(E1) \simeq 5.5 \times 10^{14} A^{2/3}\ \mathrm{sec}^{-1}$.

For example a 1^- to 0^+ transition would be E1, while a 1^+ to 0^+ transition would be M1. Imagine a 3^+ to 2^- transition. j/ℓ could be from 1 to 5, but E1 would be the way the transition went.

Photons in lepton scattering can be spacelike or timelike, as in Moeller versus Bhabha scattering.

For hadrons things get sticky, since hadrons are complex systems with a radius of about 0.8 fm, having both charge and magnetic moment distributions.

Counting Quarks in $e^+ + e^-$ scattering:

$$\frac{d\sigma}{d\Omega} = \frac{(\alpha\hbar c)^2}{4s}[1 + \cos^2\theta],\ \ s = (2E_e)^2,$$

so

$$\sigma = \frac{4\pi(\alpha\hbar c)^2}{3s}.$$

You can use this process to count quarks. Define

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$

For the quarks, we replace e by $q_i e$ in α.

Then everything cancels in the ratio except $R = \Sigma_i q_i$.

For energies less than 6 GeV we might expect $R = 2/3$ because $(2/3)^2 + 2(1/3)^2 = 6/9 = 2/3$.

But experiment gives $R = 2$, because there are 3 colors!

When we raise the energy to a value greater than the mass of the heaviest quark, t, we therefore expect

$$R = 3[3(2/3)^2 + 3(1/3)^2] = 9[4/9 + 1/9] = 5,$$

and this is what is seen. Raising the energy further does not increase R.

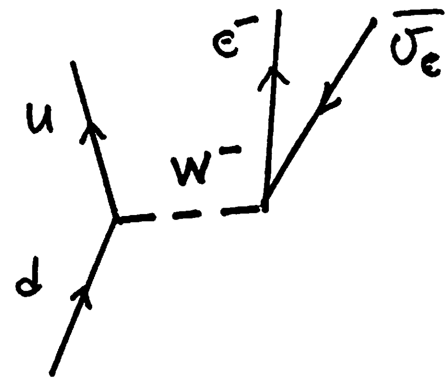
Monopoles

Axions

The Weak Interaction

$$n \to p + e^- + \bar{\nu}_e.$$

Actually of course it is



$$d \to u + e^- + \bar{\nu}_e.$$

In a nucleus,

$$^{14}_{6}O_8 \to \ ^{14}_{7}N_7 + e^+ + \nu_e, \text{ proton decay}$$

Average lifetimes are from the order of seconds to minutes, especially in nuclear processes.

Fermi's theory of the weak interaction was the first example of what is now called an "effective field theory," which basically cuts off at a certain scale, so that interactions which occur over a very, very short distance are treated as point vertexes.

If we want to use electromagnetic processes as an analogy, the immediate problem we face is that two of the three weak bosons, the ones most commonly encountered, are charged: W^+ and W^-. The other weak boson, Z^0, competes mainly with photons, and so its contributions are hard to spot.

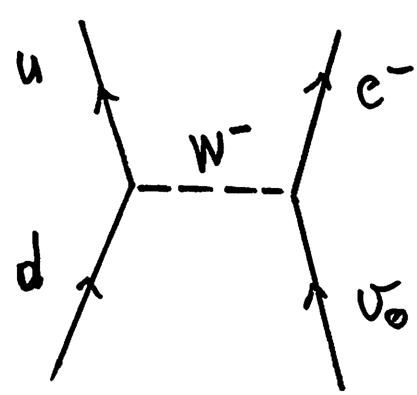
The electromagnetic analogy would be a current-current interaction. In the weak case we would have a hadronic current coupling to a weak current. The interaction is short-range because $R_w \simeq \hbar/M_W c \simeq 197/(80 \times 1000) \simeq 2.5 \times 10^{-3}$ fm.

$$H_{\mathrm{em}} = -(e^2/c^2) \int d^3\mathbf{r} d^3\mathbf{r}' \mathbf{j}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}') f(|\mathbf{r} - \mathbf{r}'|).$$

So for a weak process we will try

$$H_w = -(g_w^2/c^2) \int d^3\mathbf{r} d^3\mathbf{r}' \mathbf{J}_W^\ell(\mathbf{r}) \cdot \mathbf{J}_W^h(\mathbf{r}') f(|\mathbf{r} - \mathbf{r}'|).$$

For $f(r)$ we could take any function with short range, like a Yukawa function or a Gaussian function... something that for tiny R is practically a δ-function.



If we ignore the possible variation of the currents over R_w, then we can reduce the integral to

$$H_w = -(4\pi g_w^2 R_w^2/c^2) \int d^3\mathbf{r} \mathbf{J}_W^\ell(\mathbf{r}) \cdot \mathbf{J}_W^h(\mathbf{r}).$$

Fermi's approach introduced a constant G_F such that

$$H_w = -(G_F/(\sqrt{2}c^2)) \int d^3\mathbf{r} \mathbf{J}_W^\ell(\mathbf{r}) \cdot \mathbf{J}_W^h(\mathbf{r}).$$

But this does not describe weak interactions in general, because purely leptonic weak processes occur, and also purely hadronic. [Example: a μ^- decays to an electron, or up and down quarks interact by exchanging a W^-.]

So we write in general $\mathbf{J}_W = \mathbf{J}_W^\ell + \mathbf{J}_W^h$.

Then we might try writing something like:

$$H_w = -(G_F/(\sqrt{2}c^2)) \int d^3\mathbf{r} \mathbf{J}_W(\mathbf{r}) \cdot \mathbf{J}_W^\dagger(\mathbf{r}).$$

But we need a 4-vector current, so we can define $\mathcal{J}_W^\mu = (c\rho_W, -\mathbf{J}_W)$. Then we would have something like

$$H_W = -(G_F/(\sqrt{2}c^2)) \int d^3\mathbf{r} \mathcal{J}_W^\mu \mathcal{J}_{\mu W}^\dagger.$$

Weak processes fall into three classes:

(1) LEPTONIC: only leptons participate.

(2) SEMILEPTONIC: leptons and hadrons participate.

(3) HADRONIC: only hadrons participate. Since $\Delta S = 0$ is seen in all strong and electromagnetic processes, processes in which $|\Delta S| = 1$ are the only hadronic processes where the weak process can get a showing. [Others go far faster by strong or electromagnetic interactions.]

Let's drop the 4-vector form for now (too many subscripts and superscripts) and look again at the weak Hamiltonian.

Consider only the electron and muon for simplicity, so $\mathbf{J}_W^\ell = \mathbf{J}_W^e + \mathbf{J}_W^\mu$.

Then when we write out H_W we get e-e, e-μ, μ-e and μ-μ terms.

Now recall that we could have written

$$H_{\text{em}} = (e/c) \int d^3\mathbf{r}\mathbf{j}_{\text{em}} \cdot \mathbf{A}.$$

This would lead to

$$\langle\beta|H_{\text{em}}|\alpha\rangle = \frac{-ie\hbar}{mc} \int d^3\mathbf{r}\psi_\beta^* \nabla\psi_\alpha \cdot \mathbf{A}.$$

Thus we can identify the electromagnetic current as

$$\mathbf{j}_{\mathrm{em}} = \psi_\beta^* \mathbf{v} \psi_\alpha,$$

where $\mathbf{v} = (-i\hbar/m)\nabla$.

We can get to 4-vector form using

$$\rho_{\mathrm{em}} = \psi_\beta^* V_0 \psi_\alpha, \ \mathbf{j}_{\mathrm{em}} = c\psi_\beta^* \mathbf{V} \psi_\alpha,$$

so $\mathcal{J}_{\mathrm{em}} \to (c\rho, \mathbf{j})$. In general if $\mathcal{V} \to (V_0, \mathbf{V})$ then

$$\mathcal{J}_{\mathrm{em}} = c\psi_\beta^* \mathcal{V} \psi_\alpha.$$

Taking the EM case as our cue, we might write for first-generation weak processes,

$$\mathcal{J}_W^{e\nu} = c\psi_e^* \mathcal{V} \psi_\nu, \ \mathcal{J}_W^{e} = c\psi_e^* \mathcal{V} \psi_e,$$

$$\mathcal{J}_W^{\nu} = c\psi_\nu^* \mathcal{V} \psi_\nu.$$

Now consider what $\mathcal{P}$ does to polar vectors:

$$\mathcal{P}(V_0, \mathbf{V}) = (V_0, -\mathbf{V}).$$

But we can also have an axial vector operator, where the time-like component is a pseudoscalar instead of a scalar:

$$\mathcal{P}(\mathcal{A}_0, \vec{\mathcal{A}}) = (-\mathcal{A}_0, \vec{\mathcal{A}}).$$

It was pointed out by George Sudarshan in 1956 that to obtain the maximum possible violation of parity that was seen in weak interactions (*only* left-hand helicity states feel the weak interaction), the weak operator would have to behave like

$$\mathcal{V} - \mathcal{A}.$$

Despite Sudarshan having the idea first, the first paper in print using this approach was written by Gell-Mann and Feynman.

Consider a process in which a muon weak-decays into an electron, a muon neutrino and an electron anti-neutrino. We would expect

$$J_W^e \cdot J_W^{\mu\dagger} = c^2(\psi_e^*(\mathcal{V} - \mathcal{A})\psi_{\nu_e} \cdot \psi_{\nu_\mu}^*(\mathcal{V} - \mathcal{A})\psi_\mu).$$

So

$$W = \frac{\pi G_F^2}{\hbar}|\int d^3\mathbf{r}\psi_e^*(\mathcal{V}-\mathcal{A})\psi_{\nu_e}\cdot\psi_{\nu_\mu}^*(\mathcal{V}-\mathcal{A})\psi_\mu|^2\rho(E).$$

We can write this as

$$W = \frac{\pi G_F^2}{\hbar}|M_e - M_0|^2\rho(E),$$

where the M's are defined in the text and satisfy

$$\mathcal{P}M_e = +M_e,\ \mathcal{P}M_0 = -M_0.$$

The natural choice for the axial 4-vector is $\mathcal{A}_0 = \vec{\sigma}\cdot\mathbf{p}/(mc),\ \vec{\mathcal{A}} = \vec{\sigma}$.

Difference between chirality and helicity? Chirality is a relativistic concept, while helicity is not. For massless particles, the distinction is moot. Otherwise, the distinction involves highly complex issues.

The Fermi constant can be determined to enormous precision. Roughly, $G_F \simeq 0.9 \times 10^{-4}$ MeV-fm^3. $G_F/(\hbar c)^3 = 1.1663787 \times 10^{-5}$ GeV^{-2}.

In the most common system of units used for the weak interaction, $G_F/\sqrt{2}$ is equivalent to $g_w^2/(8M_W^2)$, so if we use about 10^{-5} for G_F, $M_W \simeq 80$ GeV, then g_W is about 0.65 and $\alpha_W = g_W^2/(4\pi) \simeq 1/30$. Therefore as we said before, and is shown on most figures comparing the α's for the three interactions include in the standard model, $\alpha_W \simeq 4.6\alpha$, in other words, the weak interaction is nearly 5 times stronger than the electromagnetic interaction.

Neutrino flavor eigenstates are mixtures of mass eigenstates and vice versa. Consider only e and μ flavors for simplicity, and suppose

$$\nu_e = \cos\theta_{12}\nu_1 + \sin\theta_{12}\nu_2,$$

$$\nu_\mu = -\sin\theta_{12}\nu_1 + \cos\theta_{12}\nu_2.$$

Then we can write

$$|\nu_e(t)\rangle = \exp(-iE_1 t/\hbar)\cos\theta_{12}|\nu_1\rangle + \exp(-iE_2 t/\hbar)\sin\theta_{12}|\nu_2\rangle.$$

Then

$$P_{\nu_\mu}(t) = |\langle\nu_\mu|\nu_e(t)\rangle|^2.$$

The result is

$$\sin^2(2\theta_{12})\sin^2[(E_1 - E_2)t/(2\hbar)].$$

Now $E_i^2 = (pc)^2 + m_i^2c^4$ and the masses are tiny compared to pc, so take the square root and expand to get $E_i \simeq pc + (m_i^2c^3/2p)$

Since the neutrinos move at almost precisely the speed of light, define the relation between time and distance travelled to be $t \simeq L/c$.

Then

$$P_{\nu_\mu}(t) \simeq \sin^2(2\theta_{12})\sin^2\left[\Delta m^2 c^3 L/(4pc\hbar)\right].$$