Fermi Gas Model of Nuclei:

We have two different species of spin-1/2 particles confined to volume V.

Thus

$$dn = \frac{4\pi p^2 dp}{(2\pi\hbar)^3} V.$$

If the system has $T = 0$ then the density of states would be

$$\frac{n}{V} = \frac{p_F^3}{6\pi^2\hbar^3}.$$

Two identical particles ($m_s = \pm(1/2)$) can be put in each state so we can write

$$\frac{N}{V} = \frac{p_{\text{Fn}}^3}{3\pi^2\hbar^3} \text{ and } \frac{Z}{V} = \frac{p_{\text{Fp}}^3}{3\pi^2\hbar^3}.$$

All nuclei basically have the same density, so that $V = (4\pi/3)R^3$ with $R = r_0 A^{1/3}$ and $r_0 \simeq 1.2$ fm.

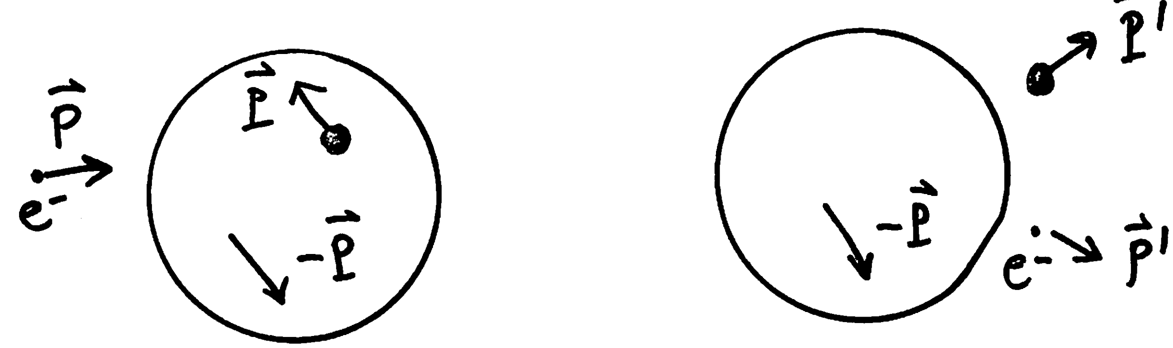
Therefore we can take $V = (4\pi/3)r_0^3 A$.

For most stable nuclei up to $A = 40$, $N \simeq Z \simeq A/2$.

Therefore

$$p_{\text{F}} = p_{\text{Fp}} = p_{\text{Fn}} = \frac{\hbar}{r_0}\left[\frac{9\pi}{8}\right]^{1/3}.$$

If we use $\hbar c = 197$ MeV-fm we find p_F is about 250 MeV/c, and quasi-elastic scattering of electrons from nuclei gives results that agree surprisingly well with this and all other predictions of the simple Fermi gas.

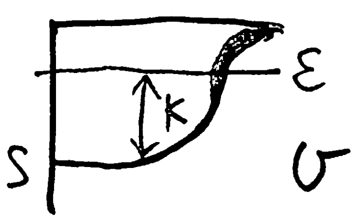


$$\vec p + \vec P = \vec p\,' + \vec P'$$

$$\vec P' = \vec q + \vec P$$

$$E + E_p = E' + E_p'$$

$$E, E' \gg m_e c^2, \quad |\vec P|, |\vec P'| \ll mc$$

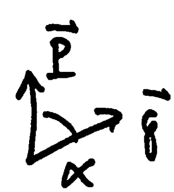


$$-\varepsilon = K - S \quad \text{so} \quad \varepsilon = S - K = S - P^2/2m$$

$$\nu = E - E' = E_p' - E_p = \left(mc^2 + P'^2/2m\right) - \left(mc^2 + \frac{P^2}{2m} - S\right)$$

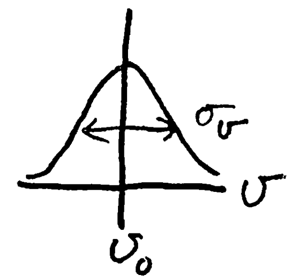
$$\text{so} \quad \nu = \frac{(\vec P + \vec q)^2}{2m} - \frac{P^2}{2m} + S$$

$$= \frac{q^2}{2m} + S - \frac{2|\vec q||\vec P|\cos\alpha}{2m}$$



It's reasonable to assume the initial nucleon momentum distribution is isotropic. Then $\nu_0 = q^2/(2m) + S$, with a width of

$$\sigma_\nu = \sqrt{\langle(\nu - \nu_0)^2\rangle} = \frac{|\mathbf{q}|}{m}\sqrt{(1/3)\langle\mathbf{P}^2\rangle}.$$



The Fermi momentum is directly related to $\langle\mathbf{P}^2\rangle$ by

$$p_F^2 = \frac{5}{3}\langle\mathbf{P}^2\rangle.$$

The Fermi energy is just $E_F = p_F^2/(2m)$ which is $(250\ \mathrm{MeV/c})^2/(2(939\ \mathrm{MeV}))$ or 33 MeV.

The typical average binding energy per nucleon, B/A, is about 8 MeV.

Therefore we can estimate V_0 for the average potential seen by a nucleon to be $V_0 = E_F + B$, which is about 41 MeV.

The average KE of a nucleon can also be estimated (accurately) in the FG picture:

$$\langle E\rangle = \frac{\int_0^{p_F} E d^3p}{\int_0^{p_F} d^3p} = \frac{3}{5}\frac{p_F^2}{2m}.$$

Table 6.1. Fermi momentum P_F and effective average potential S for various nuclei. These values were obtained from an analysis of quasi-elastic electron scattering at beam energies between 320 MeV and 500 MeV and at a fixed scattering angle of 60° [Mo71, Wh74]. The errors are approximately 5 MeV/c (P_F) and 3 MeV (S).

Nucleus	^{6}Li	^{12}C	^{24}Mg	^{40}Ca	^{59}Ni	^{89}Y	^{119}Sn	^{181}Ta	^{208}Pb
P_F [MeV/c]	169	221	235	249	260	254	260	265	265
S [MeV]	17	25	32	33	36	39	42	42	44

This gives us $\langle E \rangle$ about 20 MeV. Note that, since $\langle E \rangle = \langle P^2 \rangle/(2m)$, that $\langle P^2 \rangle$ is $(3/5)p_F^2$, as we noted earlier.

An important point is that the volume occupied by a nucleon in a nucleus is $V/A \simeq (4\pi/3)r_0^3 \simeq 7.2$ fm^3. But the volume of a single nucleon is $V_N \simeq (4\pi/3)(\langle r_p^2 \rangle^{1/2})^3$, which is only about 2.1 fm^3.

A classical model of the nucleus is not a bunch of marbles stuck together, it's more like a swarm of bees. *There is a lot of empty space.* Of course you could argue that all matter is empty space anyway, because all the particles involved as basic constituents are fundamental point particles of zero volume.

It is sometimes useful to apply the Fermi gas model to massless fermions (or fermions that are highly relativistic, so that all energies are hugely greater than mc^2 and it can be neglected).

If we write $\rho = N/V$, and use $E = pc$, then $p_F = \hbar(3\pi^2\rho)^{1/3}$, and

$$E_F = \hbar c(3\pi^2\rho)^{1/3},$$

which can be a very useful equation.

Note that $E/N = (3/4)E_F$ and $p = (1/3)(E/N)\rho$.

NUCLEAR REACTIONS!

In general we have $a + A \rightarrow b + B$. Here are some common cases... each different outgoing state and type of particle or system b is called a "reaction channel."

• ELASTIC SCATTERING: the exit particle is a, with the same center-of-momentum energy, and A is still in its original state. ["Elastic channel."]

• INELASTIC SCATTERING: the exit particle is a but it has less center-of-momentum energy, and the target nucleus A is left in an excited state, A^*. ["Inelastic channels."]

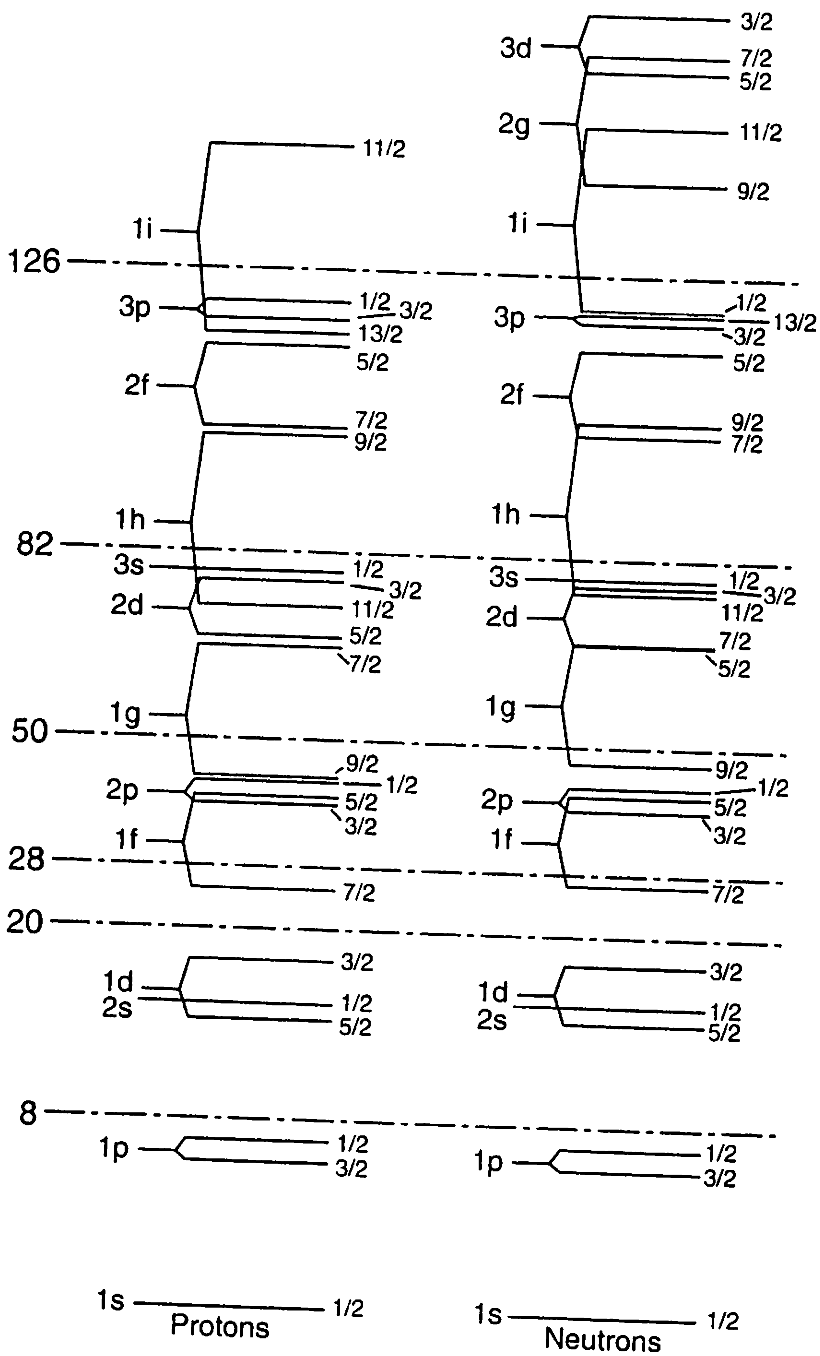
• REACTION: the exit particle b is a different particle than a, and the residual nucleus B is a different nucleus than A. For example, consider the famous (d,p) reaction. If we had, say, the process $^{40}\mathrm{Ca}(\mathrm{d},\mathrm{p})^{41}\mathrm{Ca}$, then there would be a different outgoing proton energy ("channel") for every excited state in the nucleus $^{41}\mathrm{Ca}$ that could be energetically reached during the process.

Calculations are always done in the center-of-momentum system. The vital quantity is the reaction Q-value, defined as

$$Q = [M_a + M_A - M_B - M_b]c^2.$$

With this definition the initial and final center-of-momentum energies are related by $E_f = E_i + Q$.

Because the nucleus is such a complex system, it is not unusual for different channels to be "coupled," so that what happens in one specific reaction channel (for example the reaching of a threshhold for some particular process) affects other channels.



126
82
50
28
20
8
3d
2g
1i
3p
2f
1h
3s
2d
1g
2p
1f
1d
2s
1p
1s
3/2
7/2
5/2
11/2
9/2
1/2
13/2
Protons
Neutrons

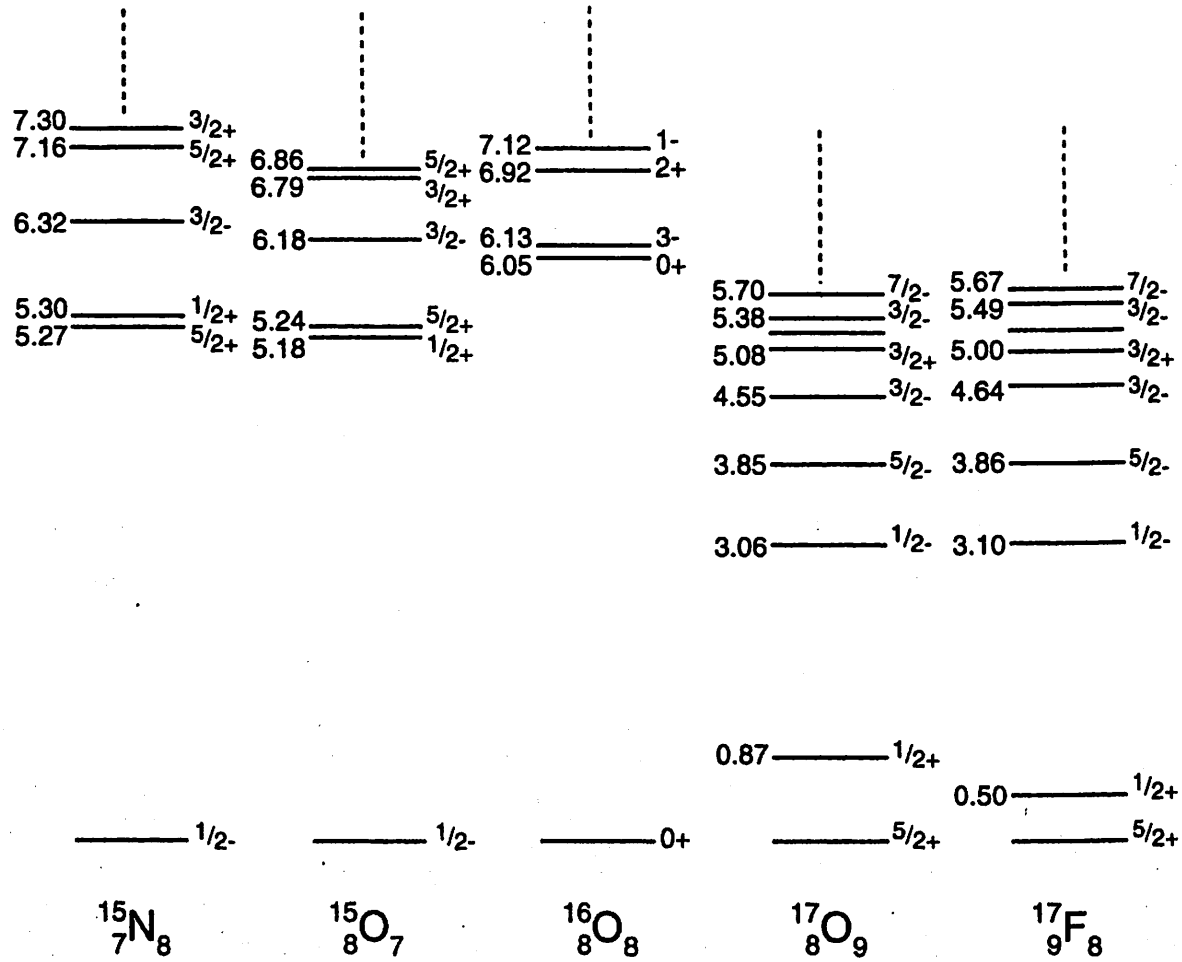
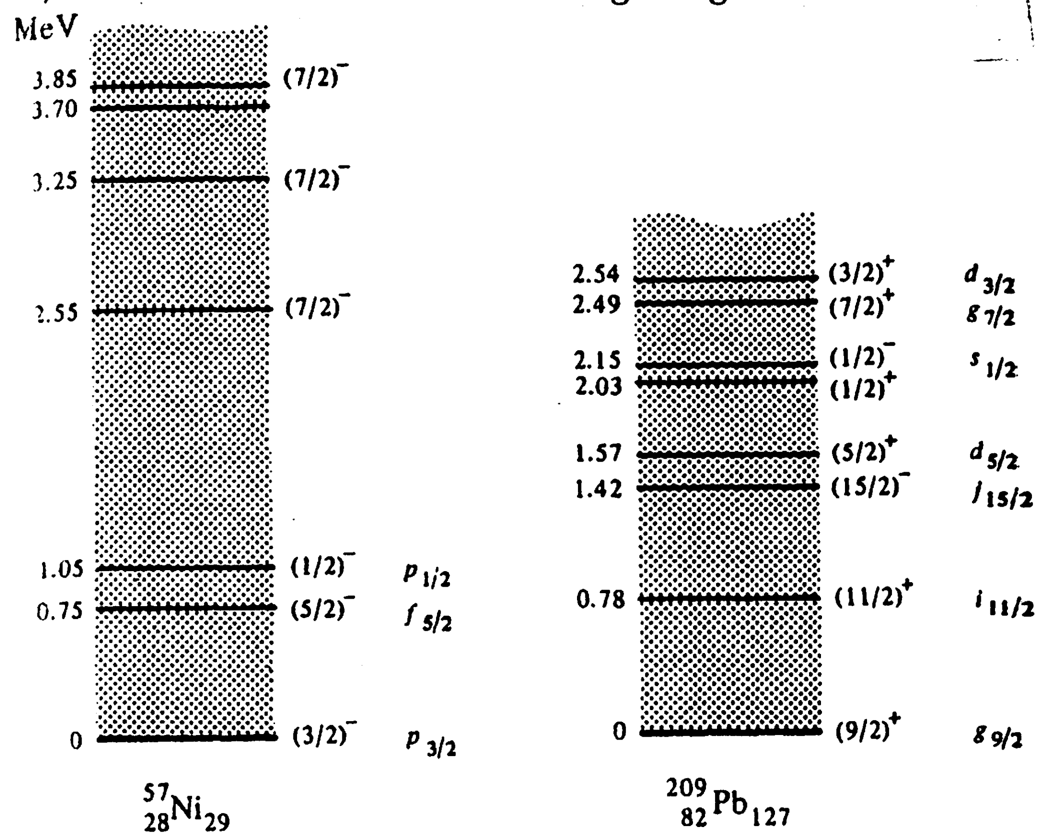


Fig. 17.7. Energy levels of the ^{15}N, ^{15}O, ^{16}O, ^{17}O and ^{17}F nuclei. The vertical axis corresponds to the excitation energy of the states with the various ground states all being set equal, i.e., the differences between the binding energies of these nuclei are not shown.



INDEPENDENT PARTICLE MODEL:

$$V(r) \Rightarrow V(r) + V_{so}(r) \ (+V_c(r) \text{ for protons}).$$

$$V(r) = \frac{-V_0}{[1+\exp((r-R_0)/a)]}, \text{ where } R_0 = r_0 A^{1/3}.$$

$$V_{so}(r) = \frac{-V_1 r_0^2}{\hbar^2 r}\frac{d}{dr}\left[\frac{1}{[1+\exp((r-R_0)/a)]}\right]\mathbf{L}\cdot\mathbf{s}.$$

$$V_c(r) = \frac{(Z-1)k_e e^2}{R_0}\left[\frac{3}{2} - \frac{r^2}{2R_0^2}\right] \text{ if } r \le R_0,$$

$$V_c(r) = \frac{(Z-1)k_e e^2}{r} \text{ if } r \ge R_0.$$

Two developments in nuclear physics coming just after the end of World War 2 hit the nuclear physics community like a thunderbolt. These were:

- **The independent particle model, aka the Shell Model.** We will discuss this shortly. It showed that many excited states of odd-A nuclei can be viewed as *one-nucleon* excitations, instead of the excitations of the entire nucleus that had been assumed to result in all nuclear states previously ("the collective model").

- **The Optical Model.** This development emphasized, because it used the earliest electronic digital computers to the fullest, that computers were going to play a major role in physics from then on.

The discovery was that elastic scattering of strongly-interacting particles from nuclei could be described in the simplest possible terms, by solving the Schrödinger equation in a partial wave expansion using a simple, complex central potential:

$$U_{\text{opt}}(r) = V(r) + iW(r),$$

where $V(r)$ was taken to have the same shape as the nuclear density. The earliest calculations were for neutron scattering. When proton scattering was to be considered, a Coulomb interaction with the

$+Ze$-charged nucleus had to be included. This was a harder calculation because the Coulomb potential results in the solution never converging to a free-particle solution, since unlike the nuclear potentials, the Coulomb potential has infinite range. However, by the late 1950s it was not hard to do such computations with available large computers.

The potential has to be complex because probability flux into the elastic channel is less than the incident probability flux, since in general many other channels are open.

Even though early calculations were primitive, they worked spectacularly well. When polarized beams of particles became available it was possible to measure the right-left asymmetry of the scattering, $(d\sigma/d\Omega)_R - (d\sigma/d\Omega)_L$, and again to describe this extremely well it was just necessary to add a spin-orbit term:

$$U_{\mathrm{opt}}(r) = V(r) + iW(r) + V_{\mathrm{so}}(r).$$

Such a term gives a different overall potential for $j = \ell + 1/2$ compared to $j = \ell - 1/2$, for (for example) an incoming proton. The result is that the potential is different for particles predominantly scattering to the left of the target nucleus, versus particles predominantly scattering to the right.

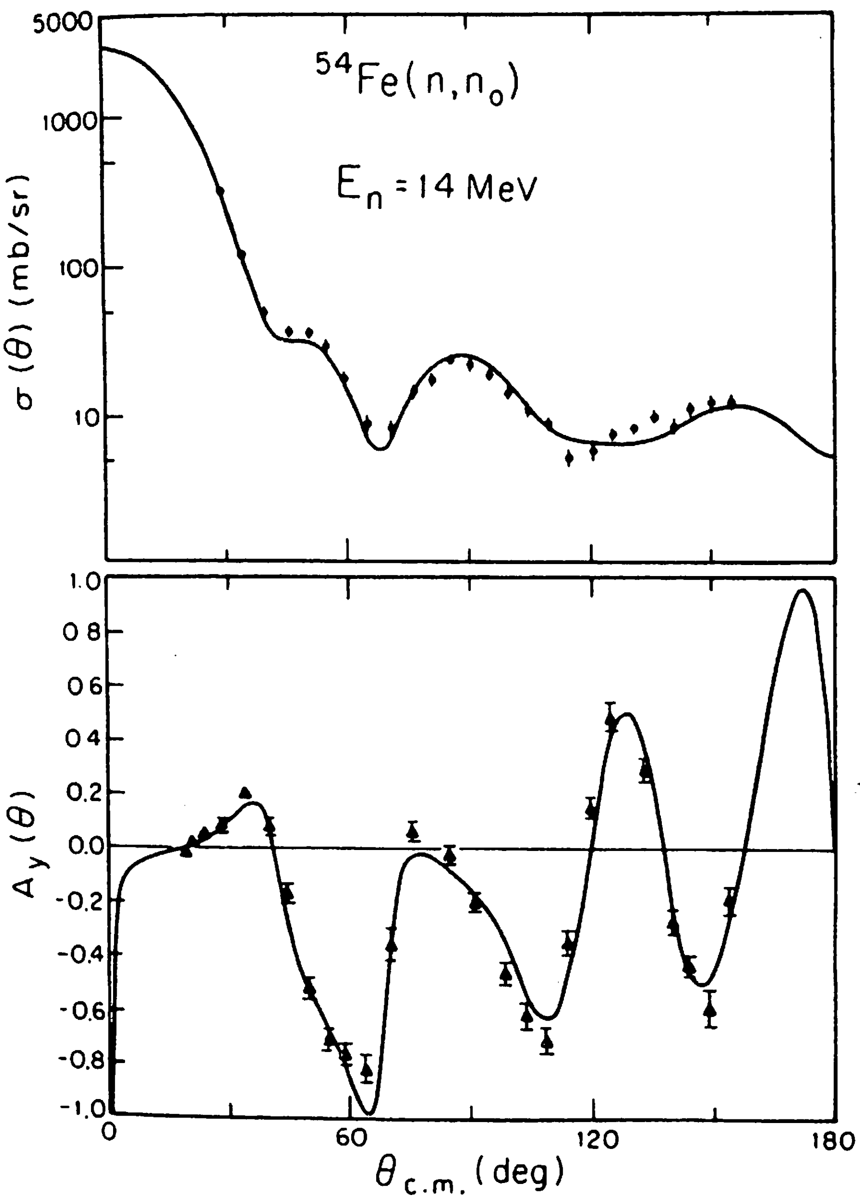


Fig. 20.5 Differential cross-sections and analysing powers for the elastic scattering of 14 MeV neutrons by ^{54}Fe compared with optical model calculations using average geometry parameters (Floyd *et al.* 1981).

Double Folding Approach for Heavy Ions

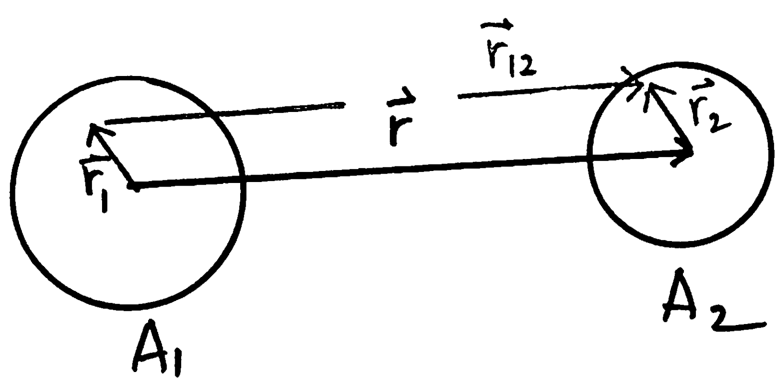
$$V(\mathbf{r}) = \int d\mathbf{r}_1 d\mathbf{r}_2 \rho_1(\mathbf{r}_1)\rho_2(\mathbf{r}_2)\nu(\mathbf{r}_{12}),$$

with $\mathbf{r}_{12} = \mathbf{r} + \mathbf{r}_2 - \mathbf{r}_1$.

This looks fearful, but remember the convolution theorem. In momentum space, this six-dimensional integral can be reduced to three one-dimensional integrals.

For spinless A_1 and A_2 the potential V depends only on r.

THE FAMOUS NUCLEAR RAINBOW...



PWBA and Direct Reactions:

Another thing that hit the nuclear physics community like a thunderbolt around 1950 was the realization that certain nuclear reactions could be realistically described using only extremely simple tools. From Fermi's Golden Rule, the cross section for a nuclear reaction should look like

$$\frac{d\sigma}{d\Omega} = \frac{m_a m_b}{(2\pi\hbar^2)^2}\frac{k_b}{k_a}|V_{fi}|^2.$$

Here, the m's are the reduced masses and the k's are the wave numbers in the incident and exit channels in the center-of-momentum system.

Without going into details of scattering and collision theory (there are many entire books on the topic) we could write the matrix element as

$$V_{\text{fi}} = \int \psi_B^\dagger \psi_b^\dagger \chi_{bB}^{(-)}(\mathbf{r}_\beta) V \psi_a \psi_A \chi_{aA}^{+}(\mathbf{r}_\alpha) d\mathbf{r}_\alpha d\mathbf{r}_\beta d\zeta,$$

where we have internal state functions (and coordinates ζ) for the systems a, A, b and B. And of course $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$ are the appropriate coordinates for the entrance and exit channel.

What people realized in about 1950 is that for a reaction, for example (d,p), with no computers available, a first try at the calculation would just involve replacing the two continuum states by plane waves, and assuming the process occurs at a specific distance R (where, for example, the neutron enters the nucleus and the proton is left as an outgoing state). Then we in effect make a zero-range approximation, and a *plane wave Born approximation*, with the integral over ζ just resulting in some constant, call it C_{bBaA}, and the only remaining coordinate being $\mathbf{r}$. The zero range approximation sets $r = R$. In other words, the initial state is taken as $\exp(+i\mathbf{k}_\alpha \cdot \mathbf{r}_\alpha)$ and the final state as $\exp(-i\mathbf{k}_\beta \cdot \mathbf{r}_\beta)$, where the zero-range approximation gives $\mathbf{r} = \mathbf{r}_\alpha = \mathbf{r}_\beta = \mathbf{R}$. The momentum transfer is $\mathbf{q} = \mathbf{k}_\alpha - \mathbf{k}_\beta$, so we wind up with:

$$V_{fi} = C_{\mathrm{bBaA}} \exp[i\mathbf{q} \cdot \mathbf{R}].$$

The next step is to make a partial wave expansion of the plane wave, to get

$$V_{fi} = C_{\mathrm{bBaA}} \sum_{\ell} i^{\ell}(2\ell + 1) j_{\ell}(qR) P_{\ell}(\cos\theta').$$

Here θ' is the angle between $\mathbf{q}$ and $\mathbf{R}$, and we can always choose the direction of $\mathbf{R}$ to make $\theta' = 0$. Where is the scattering angle θ in all of this? The magnitude of q depends directly on θ.

Note that only one term in the sum contributes. The reason is that, if the target is an even-even nucleus (which always has a 0^+ ground state), the only ℓ that contributes is the ℓ of the nuclear state the neutron goes into.

So with all these approximations the amazingly simple result is

$$\frac{d\sigma}{d\Omega} \propto |j_\ell(qR)|^2$$

Because of the distinctive peaking of the spherical Bessel function $j_\ell(z)$, the ℓ of the final state in B is almost instantly obvious. Note that $q^2 = k_\alpha^2 + k_\beta^2 - 2k_\alpha k_\beta \cos\theta$.

By 1960 it was possible to improve the calculations by using optical model solutions, rather than plane waves, for $\chi_{aA}^+(\mathbf{r}_\alpha)$ and $\chi_{bB}^+(\mathbf{r}_\beta)$. The approximation was called the distorted-wave Born approximation (DWBA), and its successes were beyond spectacular.