362L Homework set 5 **Due March 27 in class.**

["H & G" refers to our textbook. Show a *complete solution* to each question, justifying results or assumptions needed. Many of the questions are quite simple and straightforward. Do not copy stuff that you don't understand, and don't explain, out of some terse on-line collection of "solutions". Your entire grade in this class is pretty much based on homework, so take each assignment seriously.]

(1) You can use almost any E & M text as a ready reference for the following exercise. Do not just copy stuff out of the text. Present it in your own words, showing all steps. (a) Start with Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$ in vacuum, and define the usual potentials $\mathbf{A}$ and ϕ. Now make changes in the potentials using the arbitrary function $\alpha(\mathbf{r}, t)$ such that

$$\delta\phi = -(1/c)(\partial\alpha/\partial t) \text{ and } \delta\mathbf{A} = \nabla\alpha.$$

Show directly that $\delta\mathbf{E} = 0$ and $\delta\mathbf{B} = 0$.

(b) Show now that this gauge invariance leads to wave equations for $\mathbf{E}$ and $\mathbf{B}$, predicting the existence of electromagnetic radiation. In other words, gauge invariance is associated directly with a vector radiation field.

(c) A modification of Maxwell's Equations suitable for fields whose bosons have mass m would involve replacing the right-hand zero in the equation for $\nabla \cdot \mathbf{E}$ by $[-(mc^2)/\hbar^2]\phi$, and including a term $[-(mc)^2/\hbar^2]\mathbf{A}$ on the right hand side of the equation for $\nabla \times \mathbf{B}$. Show that the result of this modification is Klein-Gordon-like equations for $\mathbf{E}$ and $\mathbf{B}$, for example

$$\left(\nabla^2 - (1/c^2)(\partial^2/\partial t^2) - [(mc)^2/\hbar^2]\right)\mathbf{E} = 0.$$

As emphasized in class, terms like constant times ϕ and constant times $\mathbf{A}$ in the Maxwell Equations destroy gauge invariance.

(2) Consider a bosonic Hamiltonian $H_B = (\hbar\omega/2)(aa^\dagger + a^\dagger a)$ and show that it is equivalent to $H_B = \hbar\omega(a^\dagger a + 1/2)$, with energy eigenvalues $E_{n_B} = \hbar\omega(n_B + 1/2)$ and n_B any integer from 0 to ∞. Now consider a fermionic Hamiltonian $H_F = (\hbar\omega/2)(\alpha^\dagger\alpha - \alpha\alpha^\dagger)$, show it can be written $H_F = (\hbar\omega)(\alpha^\dagger\alpha - 1/2)$, and show its eigenvalues are $E_{n_F} = \hbar\omega(n_F - (1/2))$ where n_F can only be 0 or 1.

Now show that if we define $H_S = \hbar\omega(a^\dagger a + \alpha^\dagger\alpha)$, then the bosonic state $|n_B \geq 1, n_F = 0\rangle$ is degenerate with the fermionic state $|n_B - 1, n_F = 1\rangle$.

The generators of "supersymmetry" transformations look like $Q_S = a^\dagger\alpha$, and $Q_S^\dagger = \alpha^\dagger a$. Find the anticommutation relations obeyed by $Q_S^\dagger$ and Q_S. (They will involve H_S.) Can you show that these operators transform bosonic to fermionic states, and vice versa?

(3) 14.41 [in H & G].

(4) 14.43.

(5) Fermi Gas: Quantum physics allows us to understand easily an otherwise mystifying aspect of nature: molecules, atoms, nuclei and nucleons are essentially empty space (volumes occupied by point particles). Yet these empty spaces exert pressure when anything

squeezes them... the so-called Fermi pressure. That pressure is simple to understand from basic quantum physics, for example from the Uncertainty relations. When something reduces the volume of a quantum system, its internal energy must increase. Use the Fermi gas model for the nucleus to get the nuclear Fermi pressure. [Hint: thermodyamically, $p = -(\partial U/\partial V)_S$ for constant entropy, where U is the internal energy.] If we take a nuclear density of 0.17 nucleons per cubic fm, and a Fermi energy of about 33 MeV, plus N = Z, what is the Fermi pressure of a nucleus in MeV per cubic fm, and in bar (1 bar is standard atmospheric pressure)?

(6) 15.28.